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Topology 751
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Baire Spaces

1 Definitions and Main Results

Definition 1. *If A is a subset of a (topological) space X, the **interior** of A with respect to X is denoted A° and is defined as the union of all open sets of X that are contained in A. A has **empty interior** if A contains no open set of X other than the empty set, and we write $A^\circ = \emptyset$.*

Example 1. *In the space $\mathbb{R}$, the subset $\mathbb{Q}$ has empty interior but $[0,1]^\circ = (0,1)$.*

Definition 2. *A subset A of a topological space X is **dense** if every point of X is an adherence point of A.*

Lemma 1. *The following statements are equivalent:*

A is dense in X

$\Leftrightarrow$ *Every nonempty open set of X contains a point in A*

$\Leftrightarrow$ *If $x \in X$, every neighborhood of x has a nonempty intersection with A*

$\Leftrightarrow$ $\overline{A} = X$

Lemma 2. *A subset A has empty interior in a space X if every point of A is a limit point of A^c. That is, A has empty interior implies A^c is dense in X.*

Definition 3. *A topological space X is said to be a **Baire Space** if X satisfies the **closed Baire condition**: Given any countable collection $\{A_n\}$ of closed sets in X, each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.*

Example 2. *The space $\mathbb{Q}$ is not a Baire Space. Each singleton in $\mathbb{Q}$ is closed and has empty interior in $\mathbb{Q}$. Let (q_n) be an enumeration of the set $\mathbb{Q}$. Then $\left(\bigcup_{n=1}^{\infty}\{q_n\}\right)^\circ = \mathbb{Q}^\circ = \mathbb{Q}$ which is not empty.*

On the other hand $\mathbb{Z}_+$ is a Baire Space. Since singletons are open in $\mathbb{Z}_+$, there is no subset of $\mathbb{Z}_+$ having an empty interior, except for the empty set, so $\mathbb{Z}_+$ satisfies the closed Baire condition vacuously. The key difference between the two examples is that $\mathbb{Z}$ inherits the discrete topology from $\mathbb{R}$ but $\mathbb{Q}$ does not.

Lemma 3. *A space X is a Baire space if and only if it satisfies the **open Baire condition**: given any countable collection $\{U_n\}$ of open sets in X, each of which is dense in X, their intersection $\bigcap U_n$ is also dense in X.*

Proof. Let X be a space that satisfies the open Baire condition. Let $\{A_n\}$ be a countable collection of closed sets with empty interior in X. Then $\{A_n^c\}$ is a collection of open sets, each of which by Lemma 1 are dense in X. By assumption, $\bigcap A_n^c$ is also dense in X, that is, $\overline{\bigcap A_n^c} = X$. It follows that

$$\begin{aligned}
\emptyset &= X^c \\
&= \left(\overline{\bigcap A_n^c}\right)^c \\
&= \left(\left(\bigcap A_n^c\right)^c\right)^\circ \\
&= \left(\bigcup A_n\right)^\circ
\end{aligned}$$

□

Theorem 1. *(**Baire Category Theorem.**) If X is a compact Hausdorff space or a complete metric space, then X is a Baire Space.*

Proof. Given a countable collection $\{A_n\}$ of closed sets in X having empty interiors, we want to show that their union $\bigcup A_n$ also has an empty interior in X. So, given a nonempty open set U_0 of X, we must find a point x of U_0 that does not lie in any of the sets A_n.

Consider the first set A_1. By assumption, A_1 does not contain U_0. Therefore, we may choose a point $y \in U_0 \backslash A_1$. Regularity of X, along with the fact that A_1 is closed, enables us to choose a neighborhood U_1 of y such that

$$\begin{aligned} \overline{U_1} \cap A_1 &= \emptyset \\ \overline{U_1} &\subset U_0 \end{aligned}$$

If X is metric, we also choose U_1 small enough that its diameter is less than 1. In general, given the nonempty open set U_{n-1}, we choose a point of U_{n-1} that is not in the closed set A_n, and then we choose U_n to be a neighborhood of this point such that

$$\begin{aligned} \overline{U_n} \cap A_n &= \emptyset \\ \overline{U_n} &\subset U_{n+1} \\ diam\ U_n &< \frac{1}{n} \qquad\qquad \text{in the metric case} \end{aligned}$$

We assert that the intersection $\bigcap \overline{U_n}$ is nonempty. This occurs in 2 cases: If X is compact and Hausdorff, apply the **closed characterization of Compactness:** *every collection of closed sets with the finite intersection property has a non-empty intersection.*

If X is a complete metric space, apply the **Nested Set Theorem:** *a sequence of nonempty closed sets with vanishing diameter in a complete metric space has a nonempty intersection.*

In either case we establish the existence of a point $x \in \bigcap \overline{U_n}$. Then $x \in U_0$ because $x \in \bigcap \overline{U_n} \subset \overline{U_1} \subset U_0$. And since each $\overline{U_n}$ is disjoint from A_n, it follows that $x \notin \bigcup A_n$. This completes the proof. $\square$

Lemma 4. *Any open subspace Y of a Baire space X is itself a Baire space.*

Proof. Let $\{A_n\}$ be a countable collection of closed sets of Y that have empty interiors in Y. We show that $\bigcup A_n$ has empty interior in Y.

Let $\overline{A_n}$ be the closure of A_n in X; then $\overline{A_n} \cap Y = A_n$. The set $\overline{A_n}$ has empty interior in X. For if U is a nonempty open set of X contained in $\overline{A_n}$ then U

must intersect A_n because it contains an adherence point of A_n, and therefore a point of A_n, since U is open. Thus $U \cap Y$ is a nonempty open set of Y contained in A_n, contrary to the hypothesis.

If $\bigcup A_n$ contains the nonempty open set W of Y, then $\bigcup \overline{A_n}$ also contains W, which is open in X because Y is open in X. But each set $\overline{A_n}$ has empty interior in X, contradicting the closed Baire condition. □

Theorem 2. *Let X be a topological space and (Y,d) a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions that converges pointwise to $f(x)$ where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.*

Proof. Given a positive integer N and given $\epsilon > 0$, define

$$A_N(\epsilon) = \{x \mid d(f_n(x), f_m(x) \leq \epsilon, \ \forall n, m \geq N\}$$

Note that $A_N(\epsilon)$ is closed in X, since the set of those x for which $d(f_n(x), f_m(x)) \leq \epsilon$ is closed in X by continuity of f_n and f_m, and $A_N(\epsilon)$ is the intersection of these sets for all $n, m \geq N$.

For fixed ϵ, note that $A_1(\epsilon) \subset A_2(\epsilon) \subset \ldots$, and $\bigcup_{N\in\mathbb{N}} A_N(\epsilon) = X$. For, given $x_0 \in X$, the fact that $f_n(x_0) \to f(x_0)$ implies that the sequence $(f_n(x_0))$ is Cauchy; hence $x_0 \in A_N(\epsilon)$ for some N.

Now let

$$U(\epsilon) = \bigcup_{N\in\mathbb{N}} A_N(\epsilon)^\circ$$

We shall prove two things:

(1) $U(\epsilon)$ is open and dense in X.

(2) The function f is continuous at each point of the set

$$C = \bigcap_{n\in\mathbb{N}} U(\tfrac{1}{n})$$

The theorem follows from the fact that C must be dense in X because of the open Baire condition.

To show $U(\epsilon)$ is dense in X, it suffices to show that for any nonempty open set V of X, there is an N such that the set $V \cap A_N(\epsilon)^\circ$ is nonempty. For this purpose, we note first that for each N, the set $V \cap A_N(\epsilon)$ is closed in V, so we can represent V as a countable union of closed sets:

$$V = V \cap X = V \cap \bigcup_{n\in\mathbb{N}} A_N(\epsilon) = \bigcup_{N\in\mathbb{N}} \Big(V \cap A_N(\epsilon)\Big)$$

However, by Lemma 4, V is also a Baire space, so it can't be the case that all $V \cap A_N(\epsilon)$ have empty interior; otherwise V would also have an empty interior. Therefore, for some $M \in \mathbb{N}$, $V \cap A_M(\epsilon)$ contains some nonempty open set W of V. Because V is open in X, the set W is open in X; therefore, it is contained in $A_M(\epsilon)^\circ$.

Now we show that if $x_0 \in C$, then f is continuous at x_0. Given $\epsilon > 0$, we shall find a neighborhood W of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in W$.

First, choose k such that $\frac{1}{k} < \frac{\epsilon}{3}$. Since $x_0 \in C$, we have $x_0 \in U(\frac{1}{k})$; therefore, there is an N such that $x_0 \in A_N(\frac{1}{k})^\circ$. Finally, continuity of the function f_N enables us to choose a neighborhood W of x_0, contained in $A_N(\frac{1}{k})$, such that

i.
$$d(f_N(x), f_N(x_0)) < \tfrac{\epsilon}{3} \quad \forall x \in W$$

The fact that $W \subset A_N(\frac{1}{k})$ implies that

$$d(f_n(x), f_N(x)) \leq \tfrac{1}{k} \quad \forall n \geq N, x \in W$$

Using pointwise convergence of f_n we obtain

ii.
$$d(f(x), f_N(x)) = \lim_{n\to\infty} d(f_n(x), f_N(x)) \leq \tfrac{1}{k} < \tfrac{\epsilon}{3} \quad \forall x \in W$$

In particular, since $x_0 \in W$, we have

iii.
$$d(f(x_0), f_N(x_0)) < \tfrac{\epsilon}{3}$$

Applying triangle inequality we obtain:

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \epsilon$$

$\square$

2 Applications

1. *Let X equal the countable union $\bigcup B_n$. If X is a nonempty Baire space, at least one of the sets $\overline{B_n}$ has nonempty interior.*

   Proof. This follows from the contrapositive of the closed Baire condition: X has nonempty interior in itself, but X is the countable union of closed sets $\bigcup \overline{B_n}$, so there must be at least one such $\overline{B_n}$ that does not have empty interior. □

2. *If every point x of X has a neighborhood that is a Baire Space, then X is a Baire Space*

   Proof. Using the open Baire condition, we need to show that if $\{V_n\}$ is a collection of open dense subsets in X, then $\bigcap_{n\in\mathbb{N}} V_n$ is dense in X.

   Claim (1): Let $x \in X$ and let W be an open neighborhood of x that is a Baire space. Then $W \cap \bigcap_{n\in\mathbb{N}} V_n$ is dense in W.

   Suppose that W_0 is a nonempty open subset of W; then W_0 is also open in X, and since each V_n is dense in X it follows that $V_n \cap W_0 \neq \emptyset$. Then

   $$W_0 \cap (V_n \cap W) = V_n \cap (W \cap W_0) = V_n \cap W_0 \neq \emptyset$$

   so $V_n \cap W$ is dense in W for all n. Since W is a Baire Space,

   $$\bigcap_{n\in\mathbb{N}} V_n \cap W = W \cap \bigcap_{n\in\mathbb{N}} V_n$$

   is also dense in W.

   Claim (2): $\bigcap_{n\in\mathbb{N}} V_n$ is dense in X.

   Let U be a nonempty open subset of X, let $a \in U$, let W_a be an open neighborhood of a that is a Baire space, and let $U_0 = U \cap W_a$, so U_0 is a non-empty open set of W. By claim (1),

   $$U_0 \cap \left(W_a \cap \bigcap_{n\in\mathbb{N}} V_n \right) \neq \emptyset$$

   However this intersection is clearly contained in

   $$U \cap \bigcap_{n\in\mathbb{N}} V_n$$

which must therefore also be nonempty. It follows that $\bigcap_{n\in\mathbb{N}} V_n$ is dense in X, so we conclude that X is a Baire space. $\square$

Definition 4. *A G_δ set of a space X is a countable intersection of open sets of X and can be written $\cap_{n\in\mathbb{N}} G_n$.*

Observation 1. *If $\bigcap_{n\in\mathbb{N}} G_n$ is dense in $\mathbb{R}$, then each G_n is dense in $\mathbb{R}$.*

3. *If Y is a dense G_δ in X, and if X is a Baire Space, then Y is a Baire space in the subspace topology.*

Proof. Since Y is a G_δ set of X, $Y = \bigcap_{n\in\mathbb{N}} G_n$ for sets G_n which are open and dense in X. Now let $\{V_m\}$ be a countable collection of open dense subsets of Y.

Claim (1): $\bigcap_{m\in\mathbb{N}} V_m$ is dense in Y.

For each m there is an open set W_m of X with $V_m = Y \cap W_m$.

Claim (2): each W_m is dense in X.
Let U be an nonempty open subset of X. Then $U \cap Y \neq \emptyset$, which is a nonempty open subset of Y, so $V_m \cap (U \cap Y) \neq \emptyset$. But

$$V_m \cap (U \cap Y) = (Y \cap W_m) \cap (U \cap Y) = W_m \cap (U \cap Y) \subset W_m \cap U \neq \emptyset$$

This proves claim (2).

Since X is a Baire space, $\bigcap_{n,m\in\mathbb{N}} G_n \cap W_m$is dense in X. But that intersection is

$$\bigcap_{n,m\in\mathbb{N}} G_n \cap W_m = \bigcap_m \left(\bigcap_n G_n \right) \cap W_m = \bigcap_m Y \cap W_m = \bigcap_m V_m$$

Now let U be a nonempty open set of Y. Then there exists an open set U' of X such that $U' \cap Y = U$. But

$$\emptyset \neq U' \cap \bigcap_m V_m = U' \cap \left(Y \bigcap W_m \right) = \left(U' \cap Y \right) \cap \left(Y \bigcap W_m \right) = U \cap \bigcap_m V_m$$

. This proves claim (1), so Y is a Baire space. $\square$

4. The irrationals with the subspace topology are a Baire space

Proof. The irrationals are a dense G_δ set of $\mathbb{R}$: $\bigcap_{q\in\mathbb{Q}} \mathbb{R}\backslash\{q\}$. Since $\mathbb{R}$ is a complete metric space, it is a Baire space by the Baire Category theorem. By (3), it follows that the irrationals are a Baire space. □

Definition 5. *Let $f : X \to Y$ where X is a topological space and Y is a metric space. The **oscillation of f** is defined at each $x \in X$ by*

$$\omega_f(x) = inf\{diam(f(U))|U \text{ is an open set containing } x\}$$

Specifically, if $f : X \to \mathbb{R}$ is a real-valued function on a metric space, then the oscillation is

$$\omega_f(x) = \lim_{\delta\to 0} diam(f(B(x;\delta)))$$

Lemma 5. *A function f is continuous at a point x_0 if and only if the oscillation is zero.*

Definition 6. *An F_σ set of a space X is a countable union of closed sets of X and can be written $\bigcup_{n\in\mathbb{N}} F_n$.*

Observation 2. *The complement of an F_σ set in $\mathbb{R}$ is a G_δ set.*

5. *If $f : \mathbb{R} \to \mathbb{R}$, then the set C(f) of points at which f is continuous is a G_δ set in $\mathbb{R}$.*

Proof. First I claim that the set of discontinuities of f is an F_σ set. Define

$$F_n := \{x : \omega_f(x) \geq \tfrac{1}{n}\}$$

Then

$$D(f) = \bigcup_{n\in\mathbb{N}} F_n$$

Each set F_n is closed: If x is a limit point of F_n, it is enough to show $x \in F_n$. If $\delta > 0$, $B(x;\delta) \cap F_n \neq \emptyset$, so there exists $a \in B(x;\delta)$ such that $\omega_f(a) \geq \frac{1}{n}$. However a is contained in a smaller interval, say of radius r, such that

$$\tfrac{1}{n} \leq \operatorname{diam}(f(B(a;r))) \leq \operatorname{diam}(f(B(x;\delta)))$$

Since δ was arbitrary, we have

$$\lim_{\delta\to 0} \operatorname{diam}(f(B(x;\delta))) \geq \tfrac{1}{n} \Rightarrow \omega_f(x) \geq \tfrac{1}{n} \Rightarrow x \in F_n$$

so F_n is closed. Since $D(f)$ is the countable union of closed sets, it is an F_σ set of $\mathbb{R}$. Thus $\mathbb{R}\backslash D(f) = C(f)$ is a G_δ set by observation 2. □

6. *In $\mathbb{R}$, any G_δ set where each G_n is dense in $\mathbb{R}$ must be uncountable*

Proof. Suppose $G = \bigcap_{n\in\mathbb{N}} G_n$ is a countable G_δ set of $\mathbb{R}$ where each G_n is dense in $\mathbb{R}$. Then G can be enumerated as $\{x_1, x_2, \dots\}$. Then The sets $G_n\backslash\{x_n\}$ are also open and dense in $\mathbb{R}$ for each n. Then $G' = \bigcap_{n\in\mathbb{N}} G_n\backslash\{x_n\} = \emptyset$. However G' is the countable intersection of open dense sets. Since $\mathbb{R}$ is a Baire space, G' should be dense in $\mathbb{R}$, which is a contradiction. □

Theorem 3. *There is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous precisely on a countable dense subset of $\mathbb{R}$*

Proof. Follows directly from (5) and (6). □

Theorem 4. *If (f_n) is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that (f_n) converges pointwise to a function f on $\mathbb{R}$, then C(f) is uncountable.*

Proof. By Theorem 2, $C(f)$ is dense in $\mathbb{R}$ and by (5), $C(f)$ is a G_δ set in $\mathbb{R}$. By Observation 1 and (6), $C(f)$ must be uncountable. □

References:

Munkres Topology, chapter 8: Baire Spaces
Carothers, Real Analysis; chapter 9: Category
Quora
Wikipedia